



TITLE:

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# Multiple interior layers of solutions to elliptic Sine-Gordon type ODE

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## 1 Introduction

We consider the perturbed elliptic Sine-Gordon equation on an interval

$$\begin{aligned} -u''(t) + \lambda \sin u(t) &= \mu f(u(t)), \quad u(t) > 0 \quad t \in I := (-T, T), \\ u(\pm T) &= 0, \end{aligned} \quad (1.1)$$

where  $\lambda, \mu > 0$  are parameters and  $T > 0$  is a constant. Throughout this paper, we assume:

- (A.1)  $f$  is locally Lipschitz continuous, odd in  $u$ . Furthermore,  $f(u) > 0$  for  $u > 0$ .
- (A.2) There exist constants  $C > 0$  and  $p > 1$  such that  $|f(u)| \leq C(1 + |u|^p)$  for  $u \in \mathbf{R}$ .
- (A.3)  $f(u) \leq Cu$  for  $0 < u \ll 1$ , where  $C > 0$  is a constant.
- (A.4) There exists a constant  $m > 1$  such that for  $u \in \mathbf{R}$

$$f(u)u \geq mF(u) := m \int_0^u f(s)ds.$$

The typical examples of  $f(u)$  are:

$$f(u) = |u|^{p-1}u, \quad (p > 1), \quad f(u) = |u|^{p-1}u + |u|^{q-1}u, \quad (p, q > 1).$$

The aim here is to investigate the layer structure of the solutions to (1.1) for  $\lambda \gg 1$  by using variational approach. To be more precise, we show the existence of the solutions  $u_\lambda$  which have  $2n$  multiple interior layers in  $I$  for  $\lambda \gg 1$ . The location of multiple interior layers of  $u_\lambda$  as  $\lambda \rightarrow \infty$  are also determined. Further, we show the existence of solutions  $u_\lambda$  with boundary layers.

We explain the variational framework. We consider the variational problem (M) subject to the constraint depending on  $\lambda$ :

(M) Minimize

$$L_\lambda(u) := \frac{1}{2} \int_I |u'(t)|^2 dt + \lambda \int_I (1 - \cos u(t)) dt \quad (1.2)$$

under the constraint

$$u \in M_\alpha := \left\{ u \in H_0^1(I) : K(u) := \int_I F(u(t)) dt = 2TF(\alpha) \right\}, \quad (1.3)$$

where  $\alpha > 0$  is a *fixed constant*,  $H_0^1(I)$  is the usual real Sobolev space. Then by the Lagrange multiplier theorem, we obtain solution triple  $(\lambda, \mu(\lambda), u_\lambda) \in \mathbf{R}_+^2 \times M_\alpha$  of (1.1) (and consequently  $u_\lambda \in C^2(\bar{I})$  by a standard regularity theorem) corresponding to the problem (M).

**Theorem 0 [5].** Assume (A.1)–(A.4). Let  $0 < \alpha < 2\pi$  satisfy  $F(\alpha) < F(2\pi)/2$ . Then:  
 (i)  $u_\lambda \rightarrow 2\pi$  locally uniformly on  $(-T_{\alpha,0}, T_{\alpha,0})$  as  $\lambda \rightarrow \infty$ , where  $T_{\alpha,0} := F(\alpha)T/F(2\pi)$ .  
 (ii)  $u_\lambda \rightarrow 0$  locally uniformly on  $I \setminus [-T_{\alpha,0}, T_{\alpha,0}]$  as  $\lambda \rightarrow \infty$ .  
 (iii)  $\mu(\lambda) \rightarrow 0$  as  $\lambda \rightarrow \infty$ .

We next remove the restriction  $F(\alpha) < F(2\pi)/2$  in Theorem 0. To do this, we introduce the condition (A.5.n) for a given  $n \in \mathbf{N}$ :

$$(A.5.n) \quad H(n) := F(2(n+1)\pi) - 2nF(2n\pi) + 2\sum_{k=0}^{n-1} F(2k\pi) > 0.$$

Note that "Assume (A.5.n)" implies that the assumption (A.5.n) holds only for a given  $n$ . The example of  $f$  which satisfies (A.1)–(A.5.n) for a fixed  $n \in \mathbf{N}$  is  $f(u) = |u|^{p-1}u$  for  $p > p_n$ , where  $p_n > 1$  is a constant depending on a given  $n$ .

**Theorem 1 [6].** Assume (A.1)–(A.4) and (A.5.1). Let  $0 < \alpha < 2\pi$  satisfy  $F(\alpha) \geq F(2\pi)/2$ . Then the assertions (i)–(iii) in Theorem 0 hold.

We next show the existence of the solutions  $u_\lambda$  which have  $2(n+1)$  multiple interior transition layers at  $t = \pm T_{\alpha,n}, \pm(T - T_{\alpha,n}), \pm(T - 3T_{\alpha,n}), \dots, \pm(T - (2n-1)T_{\alpha,n})$  as  $\lambda \rightarrow \infty$ , where

$$T_{\alpha,n} := (F(\alpha) - F(2n\pi))T/H(n).$$

For  $D \subset \mathbf{R}$ , let  $-D := \{-t : t \in D\} \subset \mathbf{R}$  and  $|D|$  be the Lebesgue measure of  $D$ .

**Theorem 2 [6].** Let  $n \in \mathbf{N}$  be given. Assume (A.1)–(A.4) and (A.5.n). If  $\alpha$  satisfies  $2n\pi < \alpha < 2(n+1)\pi$  and

$$F(2n\pi) < F(\alpha) < \frac{1}{2(n+1)}F(2(n+1)\pi) + \frac{1}{(n+1)}\sum_{k=0}^n F(2k\pi), \quad (1.4)$$

then as  $\lambda \rightarrow \infty$ :

- (i)  $\|u_\lambda\|_\infty < 2(n+1)\pi$ .
- (ii)  $u_\lambda \rightarrow 2(n+1)\pi$  locally uniformly on  $(-T_{\alpha,n}, T_{\alpha,n})$ .
- (iii)  $u_\lambda \rightarrow 2n\pi$  locally uniformly on  $\pm(T_{\alpha,n}, T - (2n-1)T_{\alpha,n})$ .
- (iv)  $u_\lambda \rightarrow 2k\pi$  locally uniformly on  $\pm(T - (2k+1)T_{\alpha,n}, T - (2k-1)T_{\alpha,n})$  for  $k = 1, \dots, n-1$ .
- (v)  $u_\lambda \rightarrow 0$  locally uniformly on  $\pm(T - T_{\alpha,n}, T]$ .
- (vi) There exist constants  $C_1, C_2 > 0$  such that

$$\mu(\lambda) \leq C_1 \lambda e^{-C_2 \sqrt{\lambda}}. \quad (1.5)$$

Note that if (A.5.n) is satisfied, then there exists  $\alpha > 0$  which satisfies  $2n\pi < \alpha < 2(n+1)\pi$  and (1.4) for  $n$ .

We now consider the case where the condition (1.4) does not hold. Namely, we consider  $\alpha > 0$  which satisfies  $2n\pi < \alpha < 2(n+1)\pi$  and

$$\frac{1}{2(n+1)}F(2(n+1)\pi) + \frac{1}{(n+1)}\sum_{k=0}^n F(2k\pi) \leq F(\alpha). \quad (1.6)$$

In this case,  $u_\lambda$  has multiple interior layers at  $t = \pm(T - (2k-1)S_{\alpha,n})$  ( $k = 1, \dots, n+1$ ) as  $\lambda \rightarrow \infty$ , where

$$S_{\alpha,n} := \frac{(F(2(n+1)\pi) - F(\alpha))T}{(2n+1)F(2(n+1)\pi) - 2\sum_{k=0}^n F(2k\pi)}.$$

**Theorem 3 [6].** *Let  $n \in \mathbb{N}$  be given. Assume (A.1)–(A.4), (A.5.n) and (A.5.n+1). Let  $2n\pi < \alpha < 2(n+1)\pi$  satisfy (1.6). Then as  $\lambda \rightarrow \infty$ :*

- (i)  $\|u_\lambda\|_\infty \rightarrow 2(n+1)\pi$ .
- (ii)  $u_\lambda \rightarrow 2(n+1)\pi$  locally uniformly on  $(-(T - (2n+1)S_{\alpha,n}), T - (2n+1)S_{\alpha,n})$ .
- (iii)  $u_\lambda \rightarrow 2k\pi$  locally uniformly on  $\pm(T - (2k+1)S_{\alpha,n}, T - (2k-1)S_{\alpha,n})$  for  $k = 1, \dots, n$ .
- (iv)  $u_\lambda \rightarrow 0$  locally uniformly on  $\pm(T - S_{\alpha,n}, T]$ .
- (v) The formula (1.5) holds.

Finally, we show the existence of solutions which have boundary layers.

**Theorem 4 [6].** *Let  $n \in \mathbb{N}$  be given. Assume (A.1)–(A.4) and (A.5.n). If  $\alpha = 2n\pi$ , then  $\|u_\lambda\|_\infty < 2(n+1)\pi$  for  $\lambda \gg 1$  and  $u_\lambda \rightarrow 2n\pi$  locally uniformly on  $(-T, 0) \cup (0, T)$  as  $\lambda \rightarrow \infty$ .*

The idea of the proof of Theorems 2 is as follows. By using the variational characterization of  $u_\lambda$ , we find that the shape of  $u_\lambda$  for  $\lambda \gg 1$  is like step function, each height of the steps are  $2\pi$ . We first establish an estimate  $\|u_\lambda\|_\infty < 2(n+1)\pi$  for  $\lambda \gg 1$  by using (A.5.n). Then  $u_\lambda$  must cross the line  $u = 2\pi, 4\pi, \dots, 2n\pi$ . By using this fact, we secondly establish that  $|I_{\lambda,k}| \sim 2|I_{\lambda,0}|$  for  $\lambda \gg 1$ , where  $I_{\lambda,k} \subset (0, T)$  ( $k = 1, \dots, n-1$ ) are the intervals on which  $u_\lambda \rightarrow 2k\pi$  locally uniformly as  $\lambda \rightarrow \infty$ . Finally, by using an estimate  $\|u_\lambda\|_\infty < 2(n+1)\pi$ , we prove that  $|I_{\lambda,2(n+1)}| \sim |I_{\lambda,0}|$  for  $\lambda \gg 1$ . To prove Theorem 3, we show that  $|I_{\lambda,k}| \sim 2|I_{\lambda,0}|$  for  $k = 1, 2, \dots, n$  and  $\lambda \gg 1$ .

The rest of this paper is organized as follows. We introduce some fundamental lemmas in Section 2. Based on these lemmas, we prove Theorem 2 (i) for  $n = 1$  in Section 3.

## 2 Preliminaries

In this section, we introduce some fundamental lemmas. For the full proofs, we refer to [5]. We know by [2] that a solution  $u$  of (1.1) satisfies

$$u(t) = u(-t) \quad \text{for } t \in [0, T]. \quad (2.1)$$

$$u'(t) < 0 \quad \text{for } t \in (0, T], \quad (2.2)$$

$$u'(0) = 0, u(0) = \|u\|_\infty, \quad (2.3)$$

For  $0 \leq r \leq \|u_\lambda\|_\infty$ , let  $t_{r,\lambda} \in [0, T]$  satisfy  $u_\lambda(t_{r,\lambda}) = r$ , which exists uniquely by (2.2). The following notation will be used repeatedly. For a fixed  $0 < \epsilon \ll 1$ , let

$$l_{\lambda,\epsilon} := t_{2\pi,\lambda} - t_{2\pi+\epsilon,\lambda}, \quad m_{\lambda,\epsilon} := t_{2\pi-\epsilon,\lambda} - t_{2\pi,\lambda}, \quad \delta_{\lambda,\epsilon} := T - t_{\epsilon,\lambda}.$$

In what follows, we always fix  $0 < \epsilon \ll 1$  first. Then let  $\lambda \rightarrow \infty$ . Therefore, the standard notation  $o(1)$  will be used for  $\lambda \gg 1$ . Furthermore, the notation  $l_{\lambda,\epsilon} = \delta_{\lambda,\epsilon} + O(\epsilon) + o(1)$  (for instance) means that  $|l_{\lambda,\epsilon} - \delta_{\lambda,\epsilon}| \leq C\epsilon + o(1)$  for  $0 < \epsilon \ll 1$  fixed and  $\lambda \gg 1$ .

**Lemma 2.1** *Assume that  $(\lambda, \mu, u) \in \mathbf{R}_+ \times \mathbf{R} \times C^2(\bar{I})$  satisfies (1.1). Then  $\mu > 0$ . Further, for  $t \in \bar{I}$ ,*

$$\frac{1}{2}u'(t)^2 + \mu F(u(t)) + \lambda \cos u(t) = \frac{1}{2}u'(T)^2 + \lambda = \mu F(\|u\|_\infty) + \lambda \cos \|u\|_\infty. \quad (2.4)$$

**Proof.** Multiply the equation in (1.1) by  $u'(t)$ . Then we have

$$\frac{d}{dt} \left\{ \frac{1}{2}u'(t)^2 + \mu F(u(t)) + \lambda \cos u(t) \right\} = 0, \quad t \in \bar{I}.$$

Hence, for  $t \in \bar{I}$ ,

$$\frac{1}{2}u'(t)^2 + \mu F(u(t)) + \lambda \cos u(t) \equiv \text{constant}. \quad (2.5)$$

By putting  $t = 0, T$  in (2.5), we obtain (2.4) by (2.3). Then by (2.4), we obtain

$$\mu F(\|u\|_\infty) = \frac{1}{2}u'(T)^2 + \lambda(1 - \cos \|u\|_\infty) > 0. \quad (2.6)$$

Since  $F(\|u\|_\infty) > 0$  by (A.1),  $\mu > 0$  follows from (2.6). ■

**Lemma 2.2** *Let  $\alpha > 0$  and  $\lambda > 0$  be fixed. Then there exists  $(\mu(\lambda), u_\lambda) \in \mathbf{R}_+ \times (M_\alpha \cap C^2(\bar{I}))$  which satisfies (1.1) and  $L_\lambda(u_\lambda) = \beta(\lambda) := \inf_{u \in M_\alpha} L(u)$ .*

Lemma 2.2 can be proved easily by choosing a minimizing sequence.

**Lemma 2.3** *Let  $\alpha > 0$  be fixed. Then  $L_\lambda(u_\lambda) \leq C\lambda^{\frac{m+2}{2(m+1)}}$  for  $\lambda \gg 1$ .*

Lemma 2.3 can be proved by finding an appropriate test function  $\phi \in M_\alpha$

**Lemma 2.4** *Let  $\alpha > 0$  be fixed. Then  $\mu(\lambda) = o(\lambda)$  for  $\lambda \gg 1$ .*

Lemma 2.4 is a consequence of Lemma 2.3. By Lemma 2.3, we obtain the following (2.7). Put  $J_{\lambda,k,\delta} := \{t \in I : 2(k-1)\pi + \delta < u_\lambda(t) < 2k\pi - \delta\}$  for  $0 < \delta \ll 1$  and  $k \in \mathbf{N}$ . By Lemma 2.3, as  $\lambda \rightarrow \infty$ ,

$$\begin{aligned} |J_{\lambda,k,\delta}| &\leq \frac{1}{1 - \cos \delta} \int_{J_{\lambda,k,\delta}} (1 - \cos u_\lambda(t)) dt \\ &\leq \frac{\lambda^{-1}}{1 - \cos \delta} L_\lambda(u_\lambda) \leq C\lambda^{-m/(2(m+1))} \rightarrow 0. \end{aligned} \quad (2.7)$$

**Lemma 2.5** *Let  $\alpha > 0$  be fixed. Then  $|u'_\lambda(T)|^2/\lambda \rightarrow 0$  as  $\lambda \rightarrow \infty$ .*

Lemma 2.5 follows from (2.4) and Lemma 2.4.

**Lemma 2.6** *Let  $\alpha > 0$  and  $0 < \epsilon \ll 1$  be fixed. Then for  $\lambda \gg 1$*

$$u'_\lambda(T)^2 \leq C\lambda e^{-2\delta_{\lambda,\epsilon}\sqrt{(1-2\epsilon)\lambda}}. \quad (2.8)$$

Lemma 2.6 can be proved by (2.4) and Lemma 2.5 and the following Lemma 2.7 follows from Lemma 2.6.

**Lemma 2.7** *Let  $\alpha > 0$  and  $0 < \epsilon \ll 1$  be fixed. Assume that there exists a subsequence  $\{\lambda_j\}$  of  $\{\lambda\}$  ( $\lambda_j \rightarrow \infty$  as  $j \rightarrow \infty$ ) such that  $\|u_{\lambda_j}\|_\infty \geq 2\pi$ . Then*

$$m_{\lambda_j,\epsilon} \geq \sqrt{1-2\epsilon}\delta_{\lambda_j,\epsilon} - o(1). \quad (2.9)$$

### 3 Proof of Theorem 2 (i) for $n = 1$

**Lemma 3.1** *Assume (A.1)–(A.4). Let  $\alpha > 0$  and  $0 < \epsilon \ll 1$  be fixed. Then for  $\lambda \gg 1$*

$$u'_\lambda(T)^2 \geq C_\epsilon \lambda e^{-2\delta_{\lambda,\epsilon}\sqrt{\lambda}}. \quad (3.1)$$

**Proof.** By (1.1),

$$u''_\lambda(t) + \mu(\lambda)f(u_\lambda(t)) = \lambda \sin u_\lambda(t) \leq \lambda u_\lambda(t) \quad \text{for } t \in [t_{\lambda,\epsilon}, T].$$

By this and (2.2), we obtain

$$\frac{dS_{\lambda,2}(t)}{dt} := \frac{d}{dt} \left\{ \frac{1}{2}u'_\lambda(t)^2 + \mu(\lambda)F(u_\lambda(t)) - \frac{\lambda u_\lambda(t)^2}{2} \right\} \geq 0 \quad \text{for } t \in [t_{\lambda,\epsilon}, T].$$

This implies that  $S_{\lambda,2}(t)$  is increasing on  $[t_{\lambda,\epsilon}, T]$ . Then

$$\frac{1}{2}u'_\lambda(t)^2 + \mu(\lambda)F(u_\lambda(t)) - \frac{\lambda u_\lambda(t)^2}{2} \leq \frac{1}{2}u'_\lambda(T)^2 \quad \text{for } t \in [t_{\lambda,\epsilon}, T].$$

Then for  $t \in [t_{\lambda,\epsilon}, T]$ ,

$$-u'_\lambda(t) \leq \sqrt{u'_\lambda(T)^2 + \lambda u_\lambda(t)^2 - 2\mu(\lambda)F(u_\lambda(t))} \leq \sqrt{u'_\lambda(T)^2 + \lambda u_\lambda(t)^2}. \quad (3.2)$$

Therefore, by (3.2), we obtain

$$\begin{aligned} \delta_{\lambda,\epsilon} &= T - t_{\epsilon,\lambda} = \int_{t_{\epsilon,\lambda}}^T 1 dt \geq \int_{t_{\epsilon,\lambda}}^T \frac{-u'_\lambda(t)}{\sqrt{u'_\lambda(T)^2 + \lambda u_\lambda(t)^2}} dt \\ &= \int_0^\epsilon \frac{ds}{\sqrt{u'_\lambda(T)^2 + \lambda s^2}} = \frac{1}{\sqrt{\lambda}} \log \left( \frac{|\epsilon + \sqrt{\epsilon^2 + X_{\lambda,2}^2}|}{X_{\lambda,2}} \right) \\ &\geq \frac{1}{\sqrt{\lambda}} \log \left( \frac{2\epsilon}{X_{\lambda,2}} \right), \end{aligned}$$

where  $X_{\lambda,2} := |u'_\lambda(T)|/\sqrt{\lambda}$ . This yields (3.1). ■

**Lemma 3.2** Assume (A.1)–(A.4). Let  $\alpha > 0$  and  $0 < \epsilon \ll 1$  be fixed. Suppose that there exists a subsequence  $\{\lambda_j\}_{j=1}^\infty$  such that  $\lambda_j \rightarrow \infty$  as  $j \rightarrow \infty$  and  $\|u_{\lambda_j}\|_\infty \geq 4\pi$ . Then

$$u'_{\lambda_j}(t_{2\pi, \lambda_j})^2 \leq C\lambda_j e^{-2l_{\lambda_j, \epsilon} \sqrt{(1-\epsilon)\lambda_j}}, \quad (3.3)$$

$$t_{4\pi-\epsilon, \lambda_j} - t_{4\pi, \lambda_j} \geq \sqrt{(1-\epsilon)l_{\lambda_j, \epsilon}} - o(1). \quad (3.4)$$

Lemma 3.2 can be proved by the similar arguments as those used to prove Lemma 2.6.

**Lemma 3.3** Assume (A.1)–(A.4). Let  $\alpha > 0$  and  $0 < \epsilon \ll 1$  be fixed. Suppose that there exists a subsequence  $\{\lambda_j\}$  such that  $\lambda_j \rightarrow \infty$  as  $j \rightarrow \infty$ , and  $\|u_{\lambda_j}\|_\infty \geq 2\pi$ . Then

$$u'_{\lambda_j}(t_{2\pi, \lambda_j})^2 \leq C\lambda_j e^{-2m_{\lambda_j, \epsilon} \sqrt{(1-\epsilon)\lambda_j}}. \quad (3.5)$$

**Proof.** We write  $\lambda = \lambda_j$ , for short. For  $t \in [t_{2\pi, \lambda}, t_{2\pi-\epsilon, \lambda}]$ , by (1.1),

$$\begin{aligned} u''_\lambda(t) + \mu(\lambda)f(u_\lambda(t)) &= \lambda \sin u_\lambda(t) = -\lambda \sin(2\pi - u_\lambda(t)) \\ &\leq -\lambda(1-\epsilon)(2\pi - u_\lambda(t)) = \lambda(1-\epsilon)(u_\lambda(t) - 2\pi). \end{aligned} \quad (3.6)$$

Then for  $t \in [t_{2\pi, \lambda}, t_{2\pi-\epsilon, \lambda}]$ , by (2.2) and (3.6),

$$\{u''_\lambda(t) + \mu(\lambda)f(u_\lambda(t)) - \lambda(1-\epsilon)(u_\lambda(t) - 2\pi)\}u'_\lambda(t) \geq 0.$$

This implies that for  $t \in [t_{2\pi, \lambda}, t_{2\pi-\epsilon, \lambda}]$ ,

$$\frac{dS_{\lambda,4}(t)}{dt} := \frac{d}{dt} \left\{ \frac{1}{2}u'_\lambda(t) + \mu(\lambda)F(u_\lambda(t)) - \frac{1-\epsilon}{2}(u_\lambda(t) - 2\pi)^2 \right\} \geq 0.$$

So  $S_{\lambda,4}(t)$  is non-decreasing in  $[t_{2\pi, \lambda}, t_{2\pi-\epsilon, \lambda}]$ . Then for  $t \in [t_{2\pi, \lambda}, t_{2\pi-\epsilon, \lambda}]$ , we obtain

$$\frac{1}{2}u'_\lambda(t) + \mu(\lambda)F(u_\lambda(t)) - \frac{1-\epsilon}{2}(u_\lambda(t) - 2\pi)^2 \geq \frac{1}{2}u'_\lambda(t_{2\pi, \lambda})^2 + \mu(\lambda)F(2\pi),$$

which implies

$$\frac{1}{2}u'_\lambda(t)^2 \geq \frac{1}{2}u'_\lambda(t_{2\pi, \lambda})^2 + \frac{1-\epsilon}{2}(u_\lambda(t) - 2\pi)^2. \quad (3.7)$$

By (3.7) and the same calculation as those used to prove Lemma 2.6, we obtain (3.5). ■

**Lemma 3.4** Assume (A.1)–(A.4). Let  $\alpha > 0$  and  $0 < \epsilon \ll 1$  be fixed. Suppose that there exists a subsequence  $\{\lambda_j\}$  such that  $\lambda_j \rightarrow \infty$  as  $j \rightarrow \infty$ , and  $\|u_{\lambda_j}\|_\infty \geq 4\pi$ . Then

$$t_{4\pi-\epsilon, \lambda_j} - t_{4\pi, \lambda_j} \geq \sqrt{1-\epsilon}m_{\lambda_j, \epsilon} - o(1) \quad \text{for } \lambda_j \gg 1. \quad (3.8)$$

**Lemma 3.5** Assume (A.1)–(A.4). Let  $\alpha > 0$  and  $0 < \epsilon \ll 1$  be fixed. Assume that there exists a subsequence  $\{\lambda_j\}$  such that  $\lambda_j \rightarrow \infty$  as  $j \rightarrow \infty$ , and  $\|u_{\lambda_j}\|_\infty \geq 2\pi + \epsilon$ . Then

$$l_{\lambda_j, \epsilon} = t_{2\pi, \lambda_j} - t_{2\pi+\epsilon, \lambda_j} \geq \sqrt{1-2\epsilon}\delta_{\lambda_j, \epsilon} - o(1) \quad \text{for } \lambda_j \gg 1. \quad (3.9)$$

**Proof.** We abreviate  $\lambda_j$  as  $\lambda$ . For  $t \in [t_{2\pi+\epsilon,\lambda}, t_{2\pi,\lambda}]$ , by (2.4), we obtain

$$\begin{aligned} \frac{1}{2}u'_\lambda(t)^2 &\leq \frac{1}{2}u'_\lambda(T)^2 + \lambda(1 - \cos u_\lambda(t)) = \frac{1}{2}u'_\lambda(T)^2 + \lambda(1 - \cos(u_\lambda(t) - 2\pi)) \\ &\leq \frac{1}{2}u'_\lambda(T)^2 + \frac{1}{2}\lambda(u_\lambda(t) - 2\pi)^2. \end{aligned}$$

This implies

$$-u'_\lambda(t) \leq \sqrt{\lambda(u_\lambda(t) - 2\pi)^2 + u'_\lambda(T)^2}$$

for  $t \in [t_{2\pi+\epsilon,\lambda}, t_{2\pi,\lambda}]$ . Therefore,

$$l_{\lambda,\epsilon} = t_{2\pi,\lambda} - t_{2\pi+\epsilon,\lambda} \geq \int_{t_{2\pi+\epsilon,\lambda}}^{t_{2\pi,\lambda}} \frac{-u'_\lambda(t)}{\sqrt{\lambda(u_\lambda(t) - 2\pi)^2 + u'_\lambda(T)^2}} dt = \int_0^\epsilon \frac{1}{\sqrt{\lambda s^2 + u'_\lambda(T)^2}} ds.$$

By this, we easily obtain (3.9). ■

**Lemma 3.6** Assume (A.1)–(A.4). Let  $\alpha > 0$  and  $0 < \epsilon \ll 1$  be fixed. Assume that there exists a subsequence  $\{\lambda_j\}$  such that  $\lambda_j \rightarrow \infty$  as  $j \rightarrow \infty$ , and  $\|u_{\lambda_j}\|_\infty \geq 4\pi$ . Then

$$t_{4\pi-\epsilon,\lambda_j} - t_{4\pi,\lambda_j} \geq \sqrt{1 - 2\epsilon}\delta_{\lambda_j,\epsilon} - o(1) \quad \text{for } \lambda_j \gg 1. \quad (3.10)$$

**Proof of Theorem 2.1 (i) for  $n = 1$ .** We assume (A.1)–(A.4) and (A.5.1). Let  $2\pi < \alpha < 4\pi$  which satisfies (1.4) for  $n = 1$  be fixed. We assume that there exists a subsequence of  $\{\lambda\}$ , denoted by  $\{\lambda\}$  again, such that  $\lambda \rightarrow \infty$  and  $\|u_\lambda\|_\infty \geq 4\pi$ , and derive a contradiction. Let  $0 < \epsilon \ll 1$  be fixed. By (2.7), we see that as  $\lambda \rightarrow \infty$

$$|t_{\epsilon,\lambda} - t_{2\pi-\epsilon,\lambda}|, |t_{2\pi+\epsilon,\lambda} - t_{4\pi-\epsilon,\lambda}| \rightarrow 0. \quad (3.11)$$

Then by (3.11),

$$\begin{aligned} T &= T - t_{\epsilon,\lambda} + (t_{\epsilon,\lambda} - t_{2\pi-\epsilon,\lambda}) + (t_{2\pi-\epsilon,\lambda} - t_{2\pi,\lambda}) + (t_{2\pi,\lambda} - t_{2\pi+\epsilon,\lambda}) \\ &\quad + (t_{2\pi+\epsilon,\lambda} - t_{4\pi-\epsilon,\lambda}) + t_{4\pi-\epsilon,\lambda} \\ &= \delta_{\lambda,\epsilon} + l_{\lambda,\epsilon} + m_{\lambda,\epsilon} + t_{4\pi-\epsilon,\lambda} + (t_{\epsilon,\lambda} - t_{2\pi-\epsilon,\lambda}) + (t_{2\pi+\epsilon,\lambda} - t_{4\pi-\epsilon,\lambda}) \\ &= \delta_{\lambda,\epsilon} + l_{\lambda,\epsilon} + m_{\lambda,\epsilon} + t_{4\pi-\epsilon,\lambda} + o(1). \end{aligned} \quad (3.12)$$

Therefore, by (3.4), (3.12), Lemmas 3.4 and 3.6,

$$T \leq 3(t_{4\pi-\epsilon,\lambda} - t_{4\pi,\lambda}) + t_{4\pi-\epsilon,\lambda} + O(\epsilon) + o(1) \leq 4t_{4\pi-\epsilon,\lambda} + O(\epsilon) + o(1).$$

This implies that for  $\lambda \gg 1$

$$\frac{T}{4} \leq t_{4\pi-\epsilon,\lambda} + O(\epsilon) + o(1). \quad (3.13)$$

On the other hand, by Lemmas 2.7, 3.5, (3.12) and (3.13),

$$\begin{aligned} 3\delta_{\lambda,\epsilon} &\leq \delta_{\lambda,\epsilon} + m_{\lambda,\epsilon} + l_{\lambda,\epsilon} + O(\epsilon) + o(1) = T - t_{4\pi-\epsilon,\lambda} + O(\epsilon) + o(1) \\ &\leq \frac{3}{4}T + O(\epsilon) + o(1). \end{aligned}$$



This implies that for  $\lambda \gg 1$

$$\delta_{\lambda,\epsilon} \leq \frac{1}{4}T + O(\epsilon) + o(1). \quad (3.14)$$

It is clear that

$$\begin{aligned} TF(\alpha) = \sum_{k=1}^4 B_{k,\lambda,\epsilon} &:= \int_0^{T/4-C\epsilon} F(u_\lambda(t))dt + \int_{T/4-C\epsilon}^{t_{2\pi-\epsilon,\lambda}} F(u_\lambda(t))dt \\ &+ \int_{t_{2\pi-\epsilon,\lambda}}^{t_{\epsilon,\lambda}} F(u_\lambda(t))dt + \int_{t_{\epsilon,\lambda}}^T F(u_\lambda(t))dt \end{aligned} \quad (3.15)$$

By (3.11), we obtain that  $B_{3,\lambda,\epsilon} \rightarrow 0$  as  $\lambda \rightarrow \infty$ . It is clear that  $B_{4,\lambda,\epsilon} \leq C\epsilon$ . By (3.13), we see that  $T/4 - C\epsilon \leq t_{4\pi-\epsilon,\lambda}$  for  $\lambda \gg 1$ . Then by this,

$$B_{1,\lambda,\epsilon} \geq F(4\pi - \epsilon) \left( \frac{T}{4} - C\epsilon \right) \geq \frac{TF(4\pi)}{4} - C\epsilon.$$

By (3.11) and (3.14),

$$\begin{aligned} B_{2,\lambda,\epsilon} &\geq F(2\pi - \epsilon)(t_{2\pi-\epsilon,\lambda} - T/4 + C\epsilon) \\ &= F(2\pi - \epsilon)((t_{2\pi-\epsilon,\lambda} - t_{\epsilon,\lambda}) + T - \delta_{\lambda,\epsilon} - T/4 + C\epsilon) \\ &\geq \frac{TF(2\pi)}{2} - C\epsilon - o(1). \end{aligned}$$

By these inequalities and (3.15),

$$F(\alpha) \geq \frac{F(4\pi)}{4} + \frac{F(2\pi)}{2} - C\epsilon - o(1). \quad (3.16)$$

Choose  $\epsilon$  sufficiently small. Then this contradicts (1.4) for  $n = 1$ . Thus the proof is complete. ■

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